Incumbency Advantage in Markets with Network Externalities

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Abstract

This paper analyzes the effects of network externalities on an incumbent's advantage in a static duopoly model where an entrant and an incumbent strategically determine prices. A Global Games approach is used as an equilibrium refinement, where consumers receive both a public and a private signal about the entrant's quality. While a unique equilibrium is not guaranteed in all of the cases, the incumbent's advantage arises in specific cases depending on the relative precision of the signals. As an extension, I show in a model of endogenous advertisement choice that the multiple equilibria problem is resolved because the entrant prefers an advertisement level which makes the private signal precise enough to generate a unique equilibrium.

1 Introduction

Many markets such as wireless phone networks and financial exchanges exhibit consumer lock-in[1]. In this paper, I try to find the reason why we observe incumbency advantage in these markets. These markets do not have substantial switching costs like the standard examples given for lock-in, such as QWERTY keyboard and frequent flyer programs of airlines. I claim that the existence of uncertain network externalities[2] can create endogenous switching costs which creates an advantage for the first firm in the market. Consumers may not find it individually rational to buy from the incumbent even if a coordinated switch is beneficial for most consumers.

In such markets, the market structure is expected to be different than markets with a standard price competition. It is possible to have entry deterrence as in some markets with explicit switching costs, such as cable TV markets. If entry deterrence is possible how does it affect the welfare? What are the policy implications? Are there any strategies for the entrant to overcome the incumbency advantage? In this paper, I address these questions in an incomplete information framework, using two different models of network externalities.

Network externalities on the demand side cause coordination problems for the consumers because of strategic complementarities. Since there may exist many ways to coordinate, in network industry, there are potentially many equilibria. In this framework, an additional question arises: which equilibrium will be played?

The two main approaches used in the literature to handle the multiple equilibria problem are: the higher order beliefs approach by Mertens and Zamir (1985) and the global games approach by Carlsson and van Damme (1993). This paper adopts the latter as an equilibrium refinement. Existence of imperfect information not only requires the players to form beliefs about other players but it also requires them to form beliefs about other players' beliefs, and about other players' beliefs about his belief, etc. The higher order beliefs approach by Mertens and Zamir (1985) makes the analysis more realistic in the sense that it takes higher order beliefs into account.

[1] See Ramos, 2003, Competition Between Stock Exchanges: A Survey and Federal Communications Commission 15th Report, 2011

[2] Securities and exchanges exhibit network effects since trading volume increases liquidity. mobile phone networks charge lower prices for intra-network calls.

But it also complicates the equilibrium analysis because of the same reason. On the other hand, the global games approach is sophisticated enough to capture the effects of higher order beliefs but simple enough to allow tractable analysis.

In my first model, I develop a static duopoly market with network externalities where two firms (an incumbent and an entrant) simultaneously set prices. After observing a public and a private signal about the entrant's quality along with prices, the consumers update not only their beliefs about the quality, but also their beliefs about the signals received by other consumers. Using the global games approach, this updating process generates a unique equilibrium in some cases by iterative elimination of strictly dominated strategies. I show that if the private signal is more precise relative to the public signal, the market has a unique perfect Bayesian Nash Equilibrium. In this unique PBNE, the incumbent charges a higher price and receives higher market share compared to a benchmark Hotelling model without any network externalities. I conclude that network externalities magnifies the market concentration, in the sense that the market is more unbalanced with network effects. Hence there is an incumbency advantage.

In the second model, I offer a model of endogenous advertising. The entrant sets price and the level of advertisement to strategically manipulate the coordination problem of the consumers. By choosing the level of advertisement, the entrant can determine the number as well as the characteristics of the equilibria. I show that the multiplicity problem is resolved for sufficiently small advertisement costs. By using advertisement, the entrant can make the signal received by the consumers precise enough so that the consumers are coordinated on his product. Therefore, the advertisement plays the role of a coordination device.

The originality of this research stems from the fact that there is no one-to-one association between switching costs and network effects. The size of network effects is endogenous since its magnitude depends on the result of the coordination problem of consumers. The literature about pricing in markets with switching costs and network externalities is extensive but there are few papers on the market structure in existence of uncertainty and network externalities alone.

The paper is organized as follows. I start with a brief literature review to combine 2 different literature this paper belongs. In section 3, I introduce a model of network externalities followed by a short discussion about the coordination problems and the relevant equilibrium concept. In section 4, I characterize the sufficient conditions for uniqueness of the equilibrium and analyze the unique equilibrium model, then compare it to a benchmark model with no-network externalities. In section 5, I discuss the multiplicity issue and offer a model of endogenous advertisement choice.

2 Literature Review

This paper belongs to the global game literature originated by Carlsson and Van Damme (1993) as well as the network effects literature started with Katz and Shapiro (1985). This paper deals with the coordination problem associated with the network effects using "global games" perspective. Therefore, I will analyze these two branches separately.

Katz and Shapiro (1985) show that in an oligopoly, consumption externalities give rise to demand-side economies of scale which vary with consumer expectations. Their equilibrium notion is "Fulfilled Expectations Cournot Equilibrium" where consumers expectations of the network sizes are fulfilled at the equilibrium. The information structure is a simplified version of the one in this paper in the sense that consumer expectations about the network size are assumed to be fixed. They focus on the demand side of the market, comparing industry-wide externalities with firm specific network externalities in terms of output and efficiency without analyzing the pricing decision. Economides (1996) analyzes entry in a monopoly with network externalities where the externalities affect not only each firm's demand but also the total market demand. Assuming the expectations about the network size are fixed, he shows that the monopolist may have incentives to facilitate entry if the benefit from the increase in market demand with the entry is higher than potential loss in his own demand. Following Katz and Shapiro (1985), he also restricts the equilibrium to fulfilled expectations equilibrium. In the equilibrium, expected mean sales are realized. Cabral et al. (1999) show that in a durable goods market with network externalities, price increases over time if network externalities are strong enough. They have a two-period dynamic setting where differentiated consumers can make a purchase in only one of the periods. In a Perfect Bayesian Equilibrium, sales occur in the first period rather than the second because the seller sets a price lower than the expected second period price. In another dynamic model of network externalities, Doganoglu (2003) specifies conditions for the existence of a stable Markov Perfect Equilibrium in linear strategies. He finds that in equilibrium, a firm with a higher previous market share charges a higher price. He assumes that each period, consumers benefit from the previous period's network size. He finds that in the steady state, existence of network externalities generates a more competitive market compared to a market with no network externalities.

Despite the importance of network externalities in many markets, such as telecommunication networks and financial investments, little work has been done to explain price competition in such markets. Biglaiser and Cremer (2012) offers a static model which captures the incumbency advantage and the generalize it to an infinite horizon model with free entry. They define "sedentary consumers" equilibria in which consumers will only switch to a new network if they believe others will also do so. They show that incumbency advantage is limited iin the infinite horizon model. Cabral (2011) considers a dynamic model of competition between two networks. Consumers die and are replaced with a constant hazard rate, and firms compete for new consumers by offering lock-in prices. He considers equilibria in Markov strategies. He shows that larger networks set higher prices.

The second branch of literature this paper relates deals with Global Games literature. Carlsson and VanDamme (1993) define a global game as a game of incomplete information where the uncertainty arises from the payoff structure. Each player observes a signal about the actual payoff structure of the game. As the noise vanishes, they show that the unique equilibrium of the game satisfies Harsanyi and Selten's risk dominance criterion. In a binary game, they show that a rational individual will always choose the risk dominant equilibrium even if there exist other Pareto dominant equilibria. Frankel et al. (2001) generalize this result to an arbitrary number of players and actions. They prove the limit uniqueness, i.e., they prove that there exists a unique strategy that survives iterative dominance. On the other hand, Morris and Shin (2001) look at different information structures (private and public information and private information alone) and show that the unique equilibrium result of Carlsson and VanDamme (1993) holds under specific informational assumptions. In all of these papers the focus is on coordination games and on the generation of uniqueness condition. The model closest to my paper is Argenziano (2011). She analyzes a duopoly model with product differentiation and network effects in terms of efficiency. She characterizes the conditions for a unique equilibrium and she finds that the equilibrium allocation is not efficient not only because network externalities are not fully internalized but also due to the strategic pricing decision of the firms. The market shares are too balanced because the firm with the higher quality product charges higher prices.

2.1 The Model

I consider a one period model of price competition between an incumbent (I) and an entrant (E). I has been in the market at least one period before the start of the game along with the consumers. The incumbent's product quality is common knowlwdge and it is normalized to 0. The entrant's product quality is θ_E, unknown to both firms and consumers . The quality can be thought of as the degree of match between the product and the consumer, where a higher θ_E implies a better match with the consumer's taste. Therefore the products are horizontally differentiated.

2.2 Preferences & Profits

Firms are expected profit maximizers with zero marginal cost[3]. Firms' profit functions are given by

$$\begin{aligned} \Pi_E &= \lambda p_E \\ \Pi_I &= (1-\lambda)p_I \end{aligned}$$

where λ is the network size of E, and p_i is the price charged by firm $i \in \{I, E\}$

There is a continuum of risk neutral consumers of measure 1. Consumers form expectations about the quality of the entrant's product as well as the size of the network while making purchase decisions. Consumers are required to buy one of the two products. Consumers have linear preferences over the quality and network size. Let $U(E, \lambda, \theta_E)$ be the utility function of a consumer who buys from E whose quality level is θ_E and whose ex-post network size is λ. $U(I, 1-\lambda)$ is the utility of a consumer

[3] It is a realistic assumption in the sense that many industries with network externalities exhibit zero marginal cost (wireless phone networks, software, internet, etc...

who buys from I with quality 0 and ex-post network size $(1-\lambda)$ and c is the relative marginal value of quality over externality

$$\begin{aligned} U(E, \lambda, \theta_E) &= -p_E + c\theta_E + \lambda \\ U(I, 1-\lambda) &= -p_I + (1-\lambda) \end{aligned}$$

Firms and consumers observe the same public signal. Then, the firms set prices p_I and p_E simultaneously. Consumers observe prices along with a private signal and choose the product to buy. This purchase decision depends not only on prices, but also on the consumer's beliefs about E's quality and network sizes.

2.3 Beliefs and Bayesian Updating

Firms observe a noisy public signal θ_0 about E's quality. If we interpret the quality as the degree of match between the product and the consumer, then it is reasonable to assume that E has incomplete information about his own "quality". The public signal is of the form: $\theta_0 = \theta_E + \eta$ where η is normally distributed with mean 0 and standard deviation τ.

Consumers are differentiated by a private signal about E's quality. They observe two signals: the public signal and a private signal $x_i = \theta_E + \epsilon_i$ where each ϵ_i is independently normally distributed with mean 0 and standard deviation σ. I assume that consumers bought from the incumbent previously, therefore they are perfectly informed about the incumbent's quality.

Beliefs about the Quality

I consider symmetric switching strategies in the sense that the consumers switch to E's product if they believe that E's quality is higher than some level k (same for all consumers). I define symmetric switching strategy for consumers in terms of the expected quality of the entrant's product. The benchmark switching strategy models in the literature usually defines the cutoff level as the level of signal which makes the consumer indifferent between two products. But, in my model, since there exist two independent signals it is more convenient to conduct analysis in terms of posterior expectations of quality. Specifically, I will define k as the level of the entrant's expected quality which makes the consumer indifferent between E and I.

After observing the public signal θ_0 and the private signal x_i, consumer i believes that the entrant's quality θ is normally distributed with mean $\bar{\theta}$ and variance $\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$ where[4]

$$\bar{\theta} = E[\theta_E \mid x, \theta_0] = \frac{\sigma^2\theta_0 + \tau^2 x}{\sigma^2 + \tau^2} \tag{1}$$

Now, we can define the consumers' strategy as a function of expected quality of E for the consumer

Definition 1. *A pure symmetric switching strategy around k is $s(\bar{\theta})$ such that*

$$s(\bar{\theta}) = \left\{ \begin{array}{c} E \text{ if } \bar{\theta} > k \\ I \quad \text{otherwise} \end{array} \right\}$$

where $\bar{\theta}$ is the expected quality of the entrant's product.

Beliefs of a consumer about other consumers

The coordination problem analyzed in this model has two aspects of incomplete information: incomplete information about the quality of the entrant's product and incomplete information about the network sizes arising from higher order beliefs. The presence of higher order beliefs may cause tractability problems. The approach offered by Carlsson & Van Damme (1993) is rich enough to capture the effects of higher order beliefs and simple enough to allow tractable analysis.

For consumer i , consumer j's private signal x_j satisfies the following; $x_j = \theta + \epsilon_j$. Since i knows that the entrant's quality θ is normally distributed with mean $\bar{\theta}$ and variance $\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$, i believes that $x_j \backsim \mathcal{N}(\bar{\theta}\ , \frac{2\sigma^2\tau^2+\sigma^4}{\sigma^2+\tau^2})$.[5]

In accordance with the symmetric strategy, consumer i believes that player j will purchase entrant's product if his expectation about the quality is at least k, in other words, if $\frac{\sigma^2\theta_0+\tau^2 x_j}{\sigma^2+\tau^2} > k$ or, $x_j > k + \frac{\sigma^2}{\tau^2}(k - \theta_0)$

[4]See DeGroot (1970) for complete derivation.
[5]See DeGroot (1970) for complete derivation

Since the distribution of x_j is known by i, the probability of other consumers buying the entrant's product (which is also equal to the percentage of consumers buying the entrant's product (λ)) will be:

$$\lambda = 1 - \Phi\left(\frac{k + \frac{\sigma^2}{\tau^2}(k - \theta_0) - \bar{\theta}}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right) \tag{2}$$

At the cutoff k, $\bar{\theta} = k$, therefore;

$$\lambda = 1 - \Phi\left(\sqrt{\gamma}(k - \theta_0)\right)$$

where γ is defined as:

$$\gamma = \frac{\sigma^2}{\tau^4}\left(\frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2}\right)$$

Here γ measures the relative precision of the public and private signals. It is increasing in the variance of the private signal and decreasing in the variance of the public signal. A large γ implies a less precise private signal compared to the public signal. The equilibrium will depend crucially on γ and I conduct comparative statics with respect to γ to analyze the effect of the signals on the equilibrium.

3 Results

I analyze the model in two steps by backward induction. I start with the analysis of the "induced game", which is the coordination game of the consumers for a given set of prices. Then I analyze the "full game, where firms strategically choose price anticipating the later coordination game of consumers.

3.1 Induced Game

The induced game is the continuation game starting from the decision node of the consumers. Therefore, in this subsection, prices have already been announced and signals have been observed by the consumers.

Proposition 1. *The induced game has a symmetric switching strategy equilibrium around cutoff k, where k solves*

$$ck = p_E - p_I - 1 + 2\Phi\left(\sqrt{\gamma}(k - \theta_0)\right)$$

Formally, define the expected net payoff of buying entrant's product over buying the incumbent's product as $v(\bar{\theta}, k)$.

$$v(\bar{\theta}, k) = p_I - p_E + 1 + c\bar{\theta} - 2\Phi\left(\sqrt{\gamma}(k - \theta_0)\right) \tag{3}$$

The consumer will be indifferent between the two products when $\bar{\theta} = k$. Then, the value (s) of k which satisfies $v(k, k) = p_I - p_E + 1 + ck - 2\Phi\left(\sqrt{\gamma}(k - \theta_0)\right) = 0$ is the cutoff value (s) associated with the symmetric strategy aforementioned. Existence of such k's are followed by the intermediate value theorem using the fact that for any $p_I - p_E$, $v(k, k) < 0$ for k small enough and $v(k, k) > 0$ for k large enough and continuity of v. Figure 1 shows $v(k, k)$ for 2 different levels of γ,fixing the prices and the parameters.

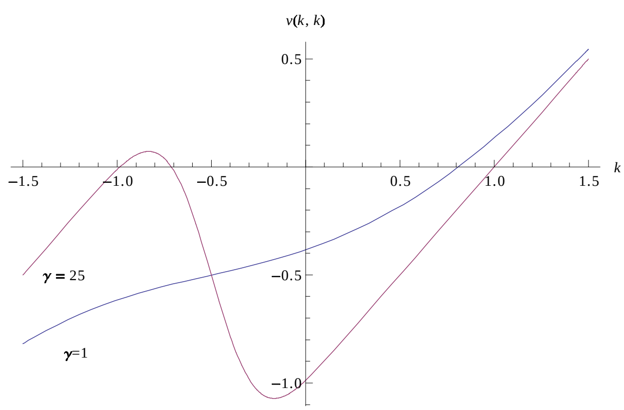

Figure 1: $v(k, k)$ for $\gamma = 1, \gamma = 25$, $\theta_0 = -0.5$, $c = 1$ and $p_I - p_E = 0$

As the figure above shows, depending on the value of γ, there may exist multiple induced game equilibria. The equilibrium cutoff levels are values where the curves hit the $x - axis$. If the private signal is precise enough (γ is small), then there exists a unique equilibrium. The next proposition gives sufficiency condition for uniqueness.

Proposition 2. *i) For any price difference $p_I - p_E$, if $\gamma < \frac{\pi c^2}{2}$, then there exists a unique k^* for which $v(k^*, k^*) = 0$ holds.*

ii) For any γ, c such that $\gamma \geq \frac{\pi c^2}{2}$, There exist price differences $p_I - p_E$ where there exist multiple induced game equilibria.

Specifically, there exist 3 equilibria iff $h(k_1) < p_I - p_E < h(k_2)$ where $k_1 < k_2$ are roots of $h'(k) = 0$ and h is defined as $h(k) = 2\Phi\left(\sqrt{\gamma}(k - \theta_0)\right) - 1 - ck$

Figure 2 shows a multiple equilibria case for $\gamma = 4$,for a fixed set of price difference and parameter values.

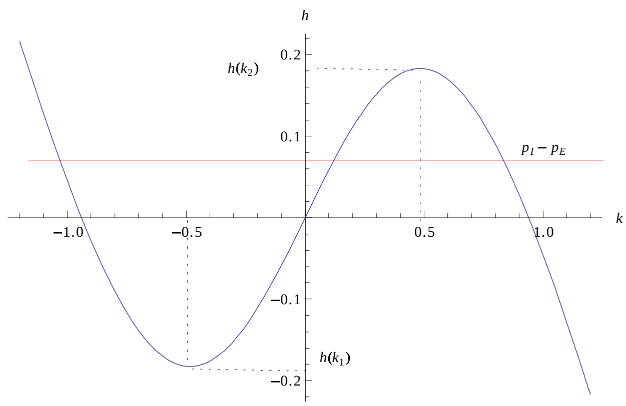

Figure 2: multiple induced game equilibria

The induced game has a unique or three equilibria[6] depending on the parameter level γ. In the next two sections, I analyzethem separetely along with some comparative statics with respect to γ.

Multiple Induced Game Equilibria

Proposition 3. *If there exist three induced game equilibria at a price difference $p_I - p_E$ for a given γ_0,[7] then there exist three induced game equilibria for any $\gamma > \gamma_0$*

Proof. Assume there exist 3 $k^{*\prime}s$ at price difference $p_I - p_E$ for a given γ_0.
By Proposition 2 , $h(k_1, \gamma_0) < p_I - p_E < h(k_2, \gamma_0)$.
Denote that $h(k_1, \gamma) < h(k_1, \gamma_0)$ and $h(k_2, \gamma) > h(k_2, \gamma_0)$ for any $\gamma > \gamma_0$.
Then, $h(k_1, \gamma_0) < p_I - p_E < h(k_2, \gamma_0)$ for all $\gamma > \gamma_0$. Finally, by Proposition 2 , there exist 3 $k^{*\prime}s$ at any $\gamma > \gamma_0$

[6] See Appendix for 2 equilibria case

[7] See Appendix for a full characterization of parameter levels γ inducing multiple equilibria

The last proposition shows the importance of the relative precision of public and private signals in coordination problems. Carlsson & Van Damme (1993) showed that even a small uncertainty about payoffs is enough to solve the multiple equilibria problem in coordination games. But, in order to reach a unique equilibrium, they assumed an information structure where there is no other coordination device such as a public signal. However in most markets, consumers not only have private signals about the quality of the products, but they may also have noisy public signals such as internet reviews, commercials or brand recognition. In my model, the induced game equilibrium depends on the relative precision of these signals. A very noisy public signal relative to the private signal (a small γ) causes the consumers to put more weight on the private signal when forming their expectations about the quality of the entrant's product. If the private signal is precise enough, then as Carlsson and Van Damme (1993) suggest, by iterated dominance the induced game will have a unique equilibrium cutoff. Conversely, a very precise public signal compared to the private signal generates possibilities of coordination at different products. Take the limiting case where $\tau = 0$, (γ will be very large). Consumers publicly observe the real quality of the entrant's product. They can be coordinated on either of the firms. If the public signal is very noisy, the weight of the private signal in the expected quality of the entrant's product will be zero. Therefore, even if a very high private signal is observed, consumers may not switch to the entrant's product, or even if a very low private signal is observed, they may still switch to the entrant.

Unique Induced Game Equilibrium

In this section, I assume that there exists a unique equilibrium of the induced game, i.e. $\gamma < \frac{\pi c^2}{2}$.

Proposition 4. *If $\gamma < \frac{\pi c^2}{2}$, for given price difference $p_I - p_E$, more people buy from the incumbent as γ increases, if initially $k > \theta_0$.*

The proposition above states that as the precision of the public signal increases, consumers value the public signal more than the private signal (they put more weight on it). Therefore, even if they receive a very high private signal, they may still decide to stay with the incumbent. Since the public signal is very accurate compared to the private signal, consumers who received a very high private signal believe that other consumers will receive lower private signals and not switch.

For example, assume the public signal is 1 and consumers decide to switch if they expect the quality to be greater than 2. In this case, a consumer who believes that the quality is 3 is going to buy from the entrant. Now, suppose they learnt that public signal is more precise than they think (τ is smaller). Now, the same consumer may decide to stay with the incumbent since right now, he believes that the public signal is more probable. Figure 3 shows the equilibrium cut-off k for different levels of γ.

If initially $k > \theta_0$, there is a private signal x associated with k such that $x > k > \theta_0$. As γ increases, the private signal becomes less precise compared to public signal. Therefore, consumers put higher weight to the public signal. Consider consumer j which observed $x_j > k > \theta_0$ and decided to buy E's product since he expects the quality to be higher than k, As γ increases, his expectation will be lower and he will buy I's product. As γ increases, consumers believe that more people will buy from incumbent since demand for I's product is $\Phi\left(\sqrt{\gamma}(k-\theta_0)\right)$. The level of k which makes the consumer indifferent between I and E will be higher[8].

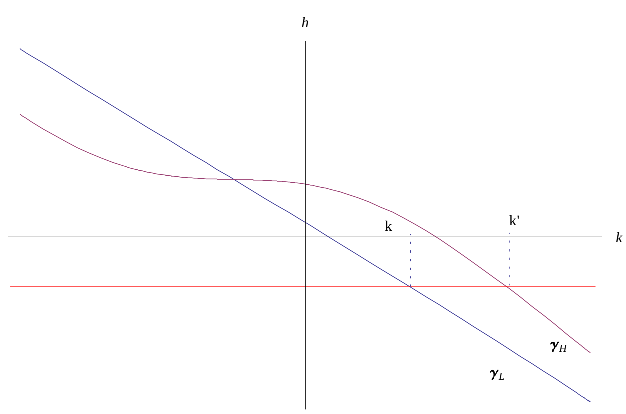

Figure 3: Induced game equilibrium for different levels of γ

[8]The equilibrium condition in the induced game of the no-network externality model is: $k = \frac{p_I - p_E}{c}$. As γ increases, k does not change. I sells more since the expected value of the quality decreases. Therefore demand to I's product will increase: $\Phi\left(\sqrt{\gamma}(k-\theta_0)\right)$

3.2 Strategic Pricing Decision

This section analyzes the price competition of the firms. Since the underlying coordination problem of the induced game may generate multiple equilibria, the full game may have a multiple equilibria issue. I start with the profit maximization problem of the firms and without making any further assumption about their beliefs, I characterize conditions for profit maximization. Once I provide the sufficient condition for uniqess, in later subsections, I will make further assumptions about consumers' beliefs.

Both E and I maximize their expected profit subject to the belief that the expected quality of E is $\bar{\theta} \sim N(\theta_0, \tau)$.

Proposition 5. *The conditions characterizing the equilibrium prices as a function of the cutoff k are*

$$\begin{aligned} 1 - \Phi\left(\frac{k-\theta_0}{\tau}\right) &= p_E \frac{\phi\left(\frac{k-\theta_0}{\tau}\right)}{\tau[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)]} \\ \Phi\left(\frac{k-\theta_0}{\tau}\right) &= p_I \frac{\phi\left(\frac{k-\theta_0}{\tau}\right)}{\tau[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)]} \end{aligned} \tag{4}$$

where k is a function of the prices.

For non-negative prices to exist, I need to assume that $c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right) > 0$ holds at the equilibrium. This condition also ensures that the law of demand[9] is satisfied.

Combining equations 3 and 4 gives the following equilibrium condition in the full game :

$$F(k^*, \gamma) = 2\Phi\left(\sqrt{\gamma}(k^* - \theta_0)\right) - ck^* - 1 - A(k^*)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k^* - \theta_0)\right)] = 0 \tag{5}$$

where $A(k) = \frac{\tau[2\Phi\left(\frac{k-\theta_0}{\tau}\right)-1]}{\phi\left(\frac{k-\theta_0}{\tau}\right)}$.

[9]By the condition for induced game equilibrium, the effect of price changes on demand is

$$\begin{aligned} \frac{\partial k}{\partial p_E} &= \frac{1}{c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)} > 0 \\ \frac{\partial k}{\partial p_I} &= -\frac{1}{c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)} < 0 \end{aligned}$$

Theorem 1 (Sufficiency Condition for Uniqueness)**.** *There exists a unique k^* and price pair (p_E^*, p_I^*) satisfying the equilibrium condition and equation 4 if $\gamma < \frac{\pi c^2}{2}$,*

Figure 4 shows the equilibrium cutoff level k^* for different levels of γ

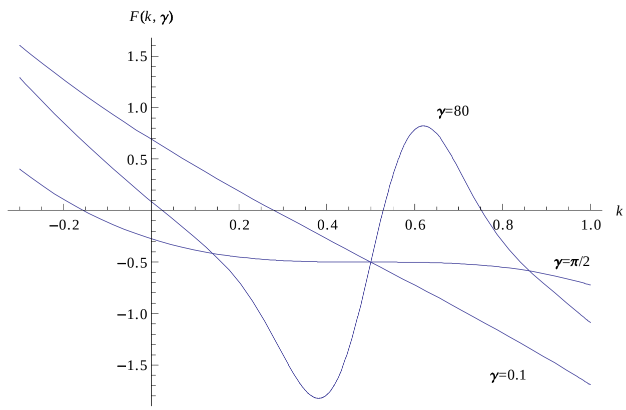

Figure 4: Equilibrium cutoff level k^* for different levels of γ

The equilibrium cutoff levels are values where the curves hit the $x - axis$.
We need to keep in mind that this unique equilibrium condition is only a sufficiency condition therefore, it does not include all possible unique equilibrium cases.
The following lemmas will be helpful in comparing equilibrium prices and in conducting comparative statics on the market shares.

Lemma 1. *For all γ, the lines $F(k, \gamma)$ intersect at the point $(\theta_0, -c\theta_0)$.*

Lemma 2. *For any γ, $F(k, \gamma)$ is symmetric around the point $(\theta_0, -c\theta_0)$*

The symmetry property[10] of the equilibrium condition ensures that analyzing one of many possible equilibria is enough to infer characteristics of other equilibria. In the next subsection, I study the unique equilibrium case using the sufficiency condition for uniqueness.

[10] The proof to Lemma 2 includes a wide treatment of the symmetry property.

Unique Full Game Equilibrium

In this section, I assume that there exists unique solution to equation 4 i.e. , $\gamma < \frac{\pi c^2}{2}$. Firms, by choosing price level, determines the equilibrium level of k. Since for any price level there exists unique k which satisfies the induced game equilibirum condition, there exists a unique k which satisfies the full equilibrium condition.

Proposition 6. *If the public signal (θ_0) is smaller (larger) than* 0*, I will have higher (lower) market share than E at the equilibrium.*

If the public signal is smaller than 0, more than half of the consumers believe that E's quality is lower than I 's quality. Therefore, even without taking the network externality into account, more than half of the consumers will buy from I. Moreover, they will expect higher network benefits if they buy from I which will make I's market share higher.

The next proposition states the effect of changes in parameter values on equilibrium statistics.

Proposition 7 (Comparative Statics)**.** *For a given γ, if the incumbent has a higher market share than the incumbent, then as γ increases,*

i) k increases and the incumbent sells more.

ii) Ratio of equilibrium prices, $\frac{p_E}{p_I}$, decreases.

iii) Incumbent charges higher prices and gets higher profits.

A higher γ implies relatively less precise private signal compared to the public signal. If I has higher market share for a given γ, then the public signal is lower than 0. As γ increases, private signal becomes less informative. The consumers put more weight to the public signal. Then more consumers will buy from I because the public signal indicates that the incumbent's quality is higher than the entrant's quality.

On the other hand, a higher market share for the incumbent implies that $p_E < p_I$.[11]. The entrant must undercut I on price in order to compete with the network advantage I has.

[11]By firms' problem: $\frac{p_E}{p_I} = \frac{1-\Phi\left(\frac{k-\theta_0}{\tau}\right)}{\Phi\left(\frac{k-\theta_0}{\tau}\right)} < 1$ if $k > \theta_0$

One of the main foci of this paper is to evaluate the effect of network externalities on the strategic pricing decision of firms. Hence it is useful to define a benchmark model without network externalities.

No-Network Externalities Model

In this benchmark model, I have the same informational assumptions as the main model except that there exists no network externalities, hence consumers do not need to form expectations about other consumers' strategies. In this case, the preferences and the profits will be the following:

$$\begin{aligned} U(E,\theta) &= -p_E + c\theta \qquad & U(I) = -p_I \\ \Pi_E &= \lambda p_E \qquad & \Pi_I = (1-\lambda)p_I \end{aligned}$$

Proposition 8. *The equilibrium cutoff level, k_{nn} satisfies the following*

$$A(k_{nn}) + k_{nn} = 0$$

where $A(k) = \frac{\tau[2\Phi\left(\frac{k-\theta_0}{\tau}\right)-1]}{\phi\left(\frac{k-\theta_0}{\tau}\right)}$

In this benchmark model, the equilibrium condition does not depend on the variance of the private signal. Consumers do not value network externalities therefore, the updating process of consumers' belief does not play a role in the consumption decision of the consumers. Naturally, expected quality depends on the private signal but its determination depends only on the public signal.

Lemma 3. *There exists a unique equilibrium price pair (p_I, p_E) and cutoff level k_{nn}. They satisfy the following:*

i) If $\theta_0 < 0$ then, $\theta_0 < k_{nn} < 0$ and I has a higher equilibrium market share and $p_E < p_I$

ii) If $\theta_0 > 0$ then, $0 < k_{nn} < \theta_0$ and E has a higher equilibrium market share and $p_I < p_E$

Without network externalities, quality is the main determinant of the market structure. If I's quality is ex-ante expected to be better than E's quality, I will charge higher prices and get higher market share compared to E.

3.3 Comparison with No-Network Externality Model

In order to isolate the effect of network externalities on the market structure, I compare the benchmark model with the network-externality model in terms of market shares and prices.

Proposition 9. *If* $\theta_0 < 0$, *Incumbent has a higher market share in the externality model compared to the no-externality case. The reverse is true if* $\theta_0 > 0$

Proof. Let the equilibrium cutoff level in the benchmark model is k_{nn}, Then,

$$A(k_{nn}) + k_{nn} = 0$$

Rearranging the equilibrium condition in the externality model

$$\begin{aligned} F(k_{nn}, \gamma) &= 2\Phi\left(\sqrt{\gamma}(k_{nn} - \theta_0)\right) - 1 - c[k_{nn} + A(k_{nn})] + 2A(k_{nn})\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k_{nn} - \theta_0)\right)] \\ &= 2\Phi\left(\sqrt{\gamma}(k_{nn} - \theta_0)\right) - 1 + 2A(k_{nn})\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k_{nn} - \theta_0)\right)] \end{aligned}$$

If $\theta_0 > 0$, by lemma 3, $k_{nn} < \theta_0$. Then,
$F(k_{nn}, \gamma) < 0$, Since F is decreasing in k, $k^* < k_{nn}$ where $F(k^*, \gamma) = 0$

The next figure shows the equilibrium cutoffs for $\theta_0 < 0$,

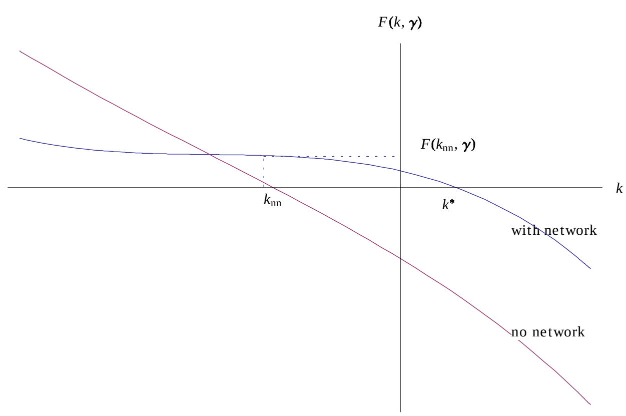

Figure 5: Comparison of market shares

Network externalities have a magnifying effect on the advantage because of the difference in quality. For instance if E's quality is expected to be lower than I's quality, even without taking network effects into account, I's market share will be higher. Network externalities, which is higher on I's side, creates an extra incentive for consumers to buy from I. Therefore, network externalities increase the assymetry in the market structure.

If there were no network externalities, an equilibrium cut-off level of $k^* = 0$ can be obtained if and only if the public signal is extremely noisy ($\tau \to \infty$). But in model with network externalities, even if τ is very large, depending on the public signal, there is be an advantage for the firm with better quality. In other words, if the public signal is higher than the incumbent's quality, existence of network externalities assures that k^* is higher than the one in the model without network externalities. The incumbent will sell more.

This proposition states that if public information signals that the entrant's quality is better than the incumbent's quality, then the consumers are willing to switch at expected qualities lower than the incumbent's quality. A slightly higher public signal can increase demand to the entrant's product a lot. This is due to the existence of network externalities.

Multiple Full Game Equilibria

In case of unique induced game equilibrium, I do not have multiple full game equilibria problem. Therefore, solving firms' profit maximization problem directly gives the equilibrium prices. For other cases where multiple equilibria may arise, I assume an additional belief system for firms to prevent issues arising from multiple equilibria. A possible multiple equilibria issue (if it exists) arises because of the coordination problem in the subgame. Each firm sets its price based on the belief that the consumers will choose the "k" that the firm wants. But actually, for any price pair, there exists 3 best response k's for consumers. In the following section, I assume that the consumers are pessimistic about other consumers' switching decision. Meaning that out of the three possibilities, each consumer believes that other consumers will choose the one which gives the entrant the lowest profit (i.e. lowerst market share). Therefore in the following section, if γ is such that there may be multiple full game equilibria, consumers are assumed to be pessimistic about other consumers' switching decision.

Firms, by choosing price level, determines the equilibrium level of k. The equilibrium selection problem arises in multiple induced game equilibria cases.The next figure illustrates this coordination issue for the entrant.

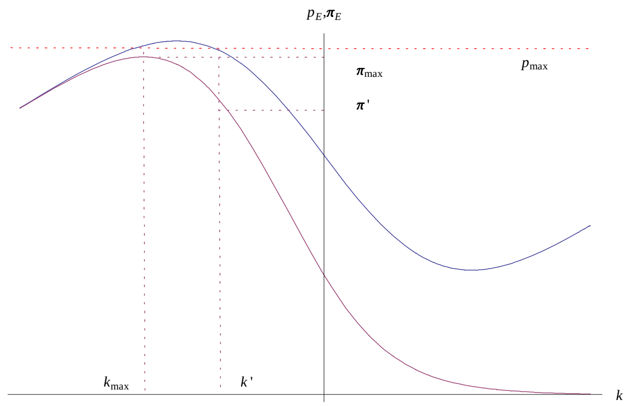

Figure 6: Coordination Problem in Full Game

As a profit maximizer, a rational entrant must choose $p_{\max}$. But consumers may respond $p_{\max}$ by playing the swiitching strategy around $k_{\max}$ or k' . If they choose $k_{\max}$, E will get the maximum profit. On the other hand, if they choose k', E's profit will be π'.

Theorem 2. *if there exist multiple equilibria for $\gamma = \gamma^*$, then there exist multiple equilibria for all $\gamma > \gamma^*$*

By the symmetry property of F, $\mid k_{\max} - \theta_0 \mid > \mid k' - \theta_0 \mid$ where $k_{\max} < k'$ are equilibrium values for a given parameter level γ. Using the FOC condition of the entrant, I know that the entrant prefers the $k_{\max}$ to k' since he gets higher profit by selling more and charging more. Therefore, an interesting question is whether there are tools which make the equilibrium the one the entrant prefers. Advertising may be such a tool if it gives the entrant the opportunity of changing the noise structure. An advertisement level which generates a γ such that there exists a unique full game equilibrium, may make E better off. In the last section, I introduce a model of endogeneous advertising.

3.4 Advertisement Decision

In this section, I assume that the entrant can undertake advertisement in order to increase the consumers' perception about his quality. Advertisement is costly but it increases the precision of the private signal. The cost of undertaking A units of advertisement is $C(A) = mA^2$ where m is a constant. Following the wireless phone network example, an advertisement can be an informative newspaper article about the characteristics of the new network. By reading this article, consumers receive more accurate information about the match of the product with their own preferences.

I will follow the same model as above except that the agents observe a private signal $x = \theta_E + \epsilon$ with $\epsilon \sim N(0, \frac{\sigma}{A})$, with $A \geq 1$. The Entrant chooses the advertisement level, A along with the price level, p_E.

Proposition 10. *If $\gamma < \frac{\pi c^2}{2}$, the entrant does not prefer to undertake advertisement if $\theta_0 > 0$.* [12]

If the private signal is precise enough so that there exists a unique equilibrium, the entrant does not want to have advertisement because the private signal (which states that E's quality is better than I) is precise enough so that most of the consumers are coordinated on E's product.

Proposition 11. *For a given γ such that there exists multiple equilibria without advertisement, If there exists an equilibrium with an advertisement level greater than 1, then $(k^* - \theta_0) < 0$, where k^* is the equilibrium cutoff level.*

In the model with multiple equilibria, there are 2 equilibria $k's$: such that $k_1 < \theta_0 < k_2$.Using the proposition above, $k^* < 0 < k_2$.[13] In other words; if consumers are pessimistic about the switching decision of other consumers, then undertaking advertisement generates higher market share for the entrant.

Proposition 12. *If $\theta_0 > 0$, for some parameter values γ such that there exist multiple equilibria without advertisement, undertaking advertisement generates higher market share to the entrant compared to the best multiple equilibria the entrant can reach.*

[12] If $\theta_0 < 0$, the entrant undertakes advertisement for sufficiently low cost of advertisement.

[13] Depending on the parameter γ and θ_0, k^* can be smaller or larger than k_1. Proposition 12 characterizes this condition on θ_0.

The next figure illustrates the proposition above. Initially, the private signal is relatively less precise compared to the public signal. Therefore there exist multiple equilibria. By undertaking advertisement, the entrant makes the private signal precise enough so that more than half of the consumers are coordinated on the entrant's product.

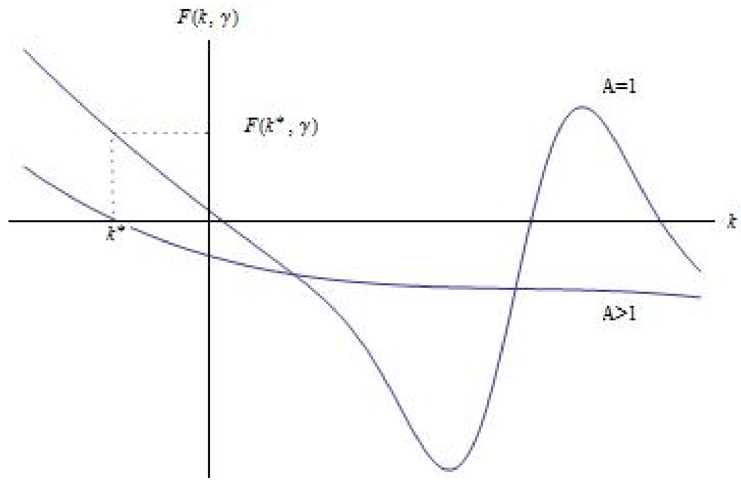

Figure 7: Unique equilibrium with advertisement.

Therefore, by using advertisement, the entrant is able to eliminate the "bad" equilibrium. The equilibrium with advertisement is better than the best equilibrium he can reach without advertisement.

4 Conclusion

In this paper, I analyze an imperfect infomation models of duopoly with network externalities on the demand side and a model of endogenous advertisement. Using a global games approach, I predict that there exists a unique equilibrium in the former if the private signal is relatively more precise compared to the public signal. Incumbency advantage arises in the sense that in the equilibrium, the incumbent charges higher price and has a higher market share compared to the entrant. In order to isolate the effect of network externalities on the equilibrium statistics, I use a benchmark duopoly case with no network externalities. The comparison shows that the network externalities have a magnifying effect on the incumbency advantage. Incumbent's price and market share is higher in the model with network externalities.

One of the main concerns in coordination games is the multiplicity of equilibria. In this paper, in cases where multiplicity is an issue, I use an additional equilibrium refinement by setting an additional assumption on the belief system of the consumers. I assume that the consumers are pessimistic about other consumers' switching decision.

In the second model, the entrant chooses the level of advertisement as well as the price level. I show that for sufficiently low costs of advertisement, the entrant chooses a costly advertisement in order to obtain a unique equilibrium rather than multiplicity. By this way, the entrant can reduce the incumbency advantage.

5 Appendix

A. Proofs of Propositions, Lemmas and Theorems

Proof of Proposition 2

i) $v(k,k)$ is strictly increasing if $\gamma < \frac{\pi c^2}{2}$, therefore there exists a unique k^* for which $v(k^*,k^*)=0$ holds.

   Formally, $v'(k,k) = c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right) > 0$ implies $\frac{c}{2\sqrt{\gamma}} > \phi\left(\sqrt{\gamma}(k-\theta_0)\right) > \frac{1}{\sqrt{2\pi}}$ since the pdf of normal distribution reaches its maximum value of $\frac{1}{\sqrt{2\pi}}$ at 0.

ii) Redefine the equilibrium condition in induced game as $p_I - p_E = 2\Phi\left(\sqrt{\gamma}(k^*-\theta_0)\right) - 1 - ck^* = h(k^*)$

   $h(k)$ has a local minimum at k_1 and a local maximum at k_2. Therefore for any $p_I - p_E$ such that $h(k_1) < p_I - p_E < h(k_2)$, There exists 3 $k^{*\prime}s$ for which the equilibrium condition holds.

   Moreover, from the symmetry of normal distribution

$$\begin{aligned} k_1 - \theta_0 &= -(k_2 - \theta_0) \\ \Phi\left(\sqrt{\gamma}(k_1-\theta_0)\right) &= 1 - \Phi\left(\sqrt{\gamma}(k_2-\theta_0)\right) \end{aligned}$$

   Then, $h(k_1) < p_I - p_E < h(k_2)$ becomes;

$$2\Phi\left(\sqrt{\gamma}(k_1-\theta_0)\right) - 1 - ck_1 < p_I - p_E < -[2\Phi\left(\sqrt{\gamma}(k_1-\theta_0)\right) - 1] - ck_2$$

Proof of Proposition 4

Using the equilibrium condition in induced game:

$$\frac{\partial k}{\partial \gamma} = -\frac{\gamma^{-1/2}(k-\theta_0)\phi\left(\sqrt{\gamma}(k-\theta_0)\right)}{2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right) - c} > 0 \ \textit{if} \ k > \theta_0$$

Incumbent sells $\Phi\left(\sqrt{\gamma}(k-\theta_0)\right)$ which increases as γ increases.

Proof of Proposition 5

Firms'problem is

$$\max_{p_E} \Pi_E = \max_{p_E} \lambda p_E$$

$$\max_{p_I} \Pi_I = \max_{p_I} (1-\lambda) p_I$$

where $\lambda = 1 - \Phi\left(\frac{k-\theta_0}{\tau}\right)$

Using IFT on equation 3

$$\begin{aligned}\frac{\partial k}{\partial p_E} &= \frac{1}{c-2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)}\\ \frac{\partial k}{\partial p_I} &= -\frac{1}{c-2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)}\end{aligned}$$

Denote that $\frac{\partial k}{\partial p_E}$ and $\frac{\partial k}{\partial p_I}$ are not continuous at k_1 and k_2 defined before as the local minimum and the local maximum of the RHS of equation 3. Therefore k is not universally continuous in prices. But, it is locally continuous on the interval $(-\infty, k_1) \cup (k_1, k_2) \cup (k_2, \infty)$.
FOCs give equation 4.

Proof of Theorem 1

$\lim_{k\to-\infty} F(k,\gamma) = \infty$ and $\lim_{k\to\infty} F(k,\gamma) = -\infty$. It is enough to prove that $F_k(k,\gamma) < 0$ for all $\dot{\gamma} < \frac{\pi c^2}{2}$.

$$F_k(k,\gamma) = 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right) - c - A'(k)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)] + A(k)[2\gamma\phi'\left(\sqrt{\gamma}(k-\theta_0)\right)]$$

using

$$\begin{aligned}\phi'(x) &= -x\phi(x)\\ A'(k) &= 2 + \frac{k-\theta_0}{\sigma^2}A(k)\end{aligned}$$

, and the fact that $A(k) > 0$ for all $k > \theta_0$ and $A(k) < 0$ for all $k < \theta_0$

$$F_k(k,\gamma) = [2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right) - c] - A'(k)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)] - A(k)[2\gamma^{3/2}(k-\theta_0)\phi\left(\sqrt{\gamma}(k-\theta_0)\right)] < 0 \text{ for all } k \text{ if } \gamma < \frac{\pi c^2}{2}$$

Then, by IVT there exists a unique k^* such that $F(k^*,\gamma) = 0$. Finally, equation 3 along with the sufficiency condition for unique equilibrium in induced game prove that
there exists unique price pair (p_E^*, p_I^*)

Proof of Lemma 1

For any γ;

$$F(\theta_0,\gamma) = 2\Phi\left(\sqrt{\gamma}(\theta_0-\theta_0)\right) - c\theta_0 - 1 - A(\theta_0)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(\theta_0-\theta_0)\right)] = -c\theta_0$$

Proof of Lemma 2

The following figure gives an overall idea about the symmetry of F.

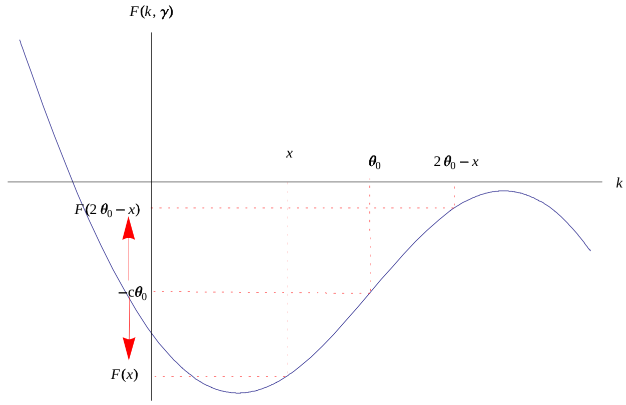

Figure 8: Symmetry of the equilibrium condition in full game

For symmetry to be satisfied, the following condition must hold:
$F(2\theta_0 - x) - (-c\theta_0) = (-c\theta_0) - F(x)$ for any x

$$F(2\theta_0 - x) = 2\Phi\left(\sqrt{\gamma}(\theta_0 - x)\right) - c(2\theta_0 - x) - 1 - A(2\theta_0 - x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(\theta_0 - x)\right)]$$

Using $\Phi(x) = 1 - \Phi(x)$ and $\phi(x) = \phi(-x)$

$$\begin{aligned}
F(2\theta_0 - x) &= 2[1 - \Phi\left(\sqrt{\gamma}(x - \theta_0)\right)] - c(2\theta_0 - x) - 1 - A(2\theta_0 - x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)] \\
&= 1 - 2\Phi\left(\sqrt{\gamma}(x - \theta_0)\right) + cx - 2c\theta_0 - A(2\theta_0 - x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)] \\
&= -[2\Phi\left(\sqrt{\gamma}(x - \theta_0)\right) - cx - 1] - 2c\theta_0 - A(2\theta_0 - x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)] \\
&= -F(x) - A(x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)] - A(2\theta_0 - x)[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)] - 2c\theta_0 \\
&= -F(x) - [c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(x - \theta_0)\right)][A(x) + A(2\theta_0 - x)] - 2c\theta_0
\end{aligned}$$

Since $A(x) = -A(2\theta_0 - x)$

$$F(2\theta_0 - x) + F(x) = -2c\theta$$

Therefore, the symmetry condition holds.

Proof of Proposition 6

Since we are still assuming that there exists a unique equilibrium, $\gamma < \frac{\pi c^2}{2}$
Equation 5 is strictly decreasing in k, and $F(\theta_0, \gamma) = -c\theta_0 < 0$
Since

$$\lim_{k \to -\infty} F(k, \gamma) = +\infty, \; k^* < \theta_0$$

Then, I 's market share is $\Phi\left(\frac{k^* - \theta_0}{\tau}\right) > 0.5$

Proof of Proposition 7

i) Using IFT,

$$\frac{\partial k}{\partial \gamma}=-\frac{\partial F/\partial \gamma}{\partial F/\partial k}=-\gamma^{-1/2}\phi[\sqrt{\gamma}(k-\theta_0)]\frac{(k-\theta_0)+A(k)[1-\gamma(k-\theta_0)^2]}{-[1+A'(k)][c-2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)]-A(k)(k-\theta_0)[2\gamma^{3/2}\phi\left(\sqrt{\gamma}(k-\theta_0)\right)]}$$

where $\partial F/\partial k<0$ by the unique equilibrium assumption
Check: $\partial F/\partial \gamma>0$
Assume $\partial F/\partial \gamma>0$ at some $k'>\theta_0$ then

$$(k'-\theta_0)+A(k')[1-\gamma(k'-\theta_0)^2]>0$$

$$\gamma<\frac{1}{A(k')(k'-\theta_0)}+\frac{1}{(k'-\theta_0)^2}$$

then for all k s.t $\theta_0<k<k', \partial F/\partial \gamma>0$
Therefore, it is enough to find a k' large enough so that our k falls within the range above.
ii) By firms' problem:

$$\frac{p_E}{p_I}=\frac{1-\Phi\left(\frac{k-\theta_0}{\tau}\right)}{\Phi\left(\frac{k-\theta_0}{\tau}\right)}$$

As γ increases, k increases, then $\frac{p_E}{p_I}$ decreases.

iii) By I's problem:

$$\Phi\left(\frac{k-\theta_0}{\tau}\right)=p_I^*\frac{\phi\left(\frac{k-\theta_0}{\tau}\right)}{\tau[c-2\sqrt{\gamma}\phi(\sqrt{\gamma}(k-\theta_0))]}$$

I need to prove that $[c-2\sqrt{\gamma}\phi(\sqrt{\gamma}(k-\theta_0))]$ increases as γ increases.
Define $Q(\gamma,k)=c-2\sqrt{\gamma}\phi(\sqrt{\gamma}(k-\theta_0))$

$$\frac{dQ(\gamma,k)}{d\gamma}=\frac{\partial Q(\gamma,k)}{\partial \gamma}+\frac{\partial Q(\gamma,k)}{\partial k}\frac{dk}{d\gamma}$$

$$\frac{dQ(\gamma,k)}{d\gamma}=\sqrt{\gamma}\phi(\sqrt{\gamma}(k-\theta_0))[-1+\gamma(k-\theta_0)^2]+2\gamma^{3/2}(k-\theta_0)\phi(\sqrt{\gamma}(k-\theta_0))\frac{dk}{d\gamma}>0$$

Proof of Proposition 8

At the equilibrium cutoff k^*,consumers will be indifferent between firms, given prices.

$$k^*=\frac{p_E-p_I}{c}$$

Firms maximization problems are the same as in the main model

$$\max_{p_E}\Pi_E=\max_{p_E}\lambda p_E$$

$$\max_{p_I}\Pi_I=\max_{p_I}(1-\lambda)p_I$$

where $\lambda=1-\Phi\left(\frac{k-\theta_0}{\tau}\right)$

except that:

$$\frac{\partial k^*}{\partial p_E} = \frac{1}{c}$$
$$\frac{\partial k}{\partial p_I} = -\frac{1}{c}$$

Therefore, the equilibrium conditions are:

$$1-\Phi\left(\frac{k-\theta_0}{\tau}\right) = p_E\frac{1}{\tau c}\phi\left(\frac{k-\theta_0}{\tau}\right)$$
$$\Phi\left(\frac{k-\theta_0}{\tau}\right) = p_I\frac{1}{\tau c}\phi\left(\frac{k-\theta_0}{\tau}\right)$$

$$k=\frac{p_E-p_I}{c}$$

Combining the equilibrium conditions below gives:

$$A(k)+k=0$$

Proof of Lemma 3

Since $A(k)$ is strictly increasing in k, there is a unique k_{nn} which satisfies the equilibrium equation.
i) Assume by contradiction, $\theta_0 < 0$ & $k_{nn} < \theta_0$, Then, $A(k_{nn}) < 0$ & $k_{nn} < 0$ which imply $A(k_{nn})+k_{nn} < 0$. Therefore, if $\theta_0 < 0, \quad k_{nn} > \theta_0$
ii) Similarly, assume $\theta_0 > 0 : k_{nn} > \theta_0$, Then $A(k_{nn}) > 0$ & $k_{nn} > 0$ which imply $A(k_{nn})+k_{nn} > 0$. Therefore, if $\theta_0 > 0, \quad k_{nn} > \theta_0$
Market shares are followed by the equilibrium conditions.

Proof of Theorem 1
If there exists multiple best response k's in induced game, then there will be multiple k's in full game.[14]

Proof of Proposition 11

$$\frac{\partial k}{\partial A} = \frac{A^{-1}\sqrt{\frac{\gamma}{A}}(k-\theta_0)\phi\left(\sqrt{\frac{\gamma}{A}}(k-\theta_0)\right)}{c-2\sqrt{\frac{\gamma}{A}}\phi\left(\sqrt{\frac{\gamma}{A}}(k-\theta_0)\right)} > 0 \text{ if } (k-\theta_0) > 0$$

therefore, $\frac{\partial \Pi_E}{\partial A} = 0$ cannot be satisfied unless $(k-\theta_0) \leq 0$

Proof of proposition 12

Suppose that k^* is a unique cutoff level generated by the advertisement choice A.

[14] See the proof in induced game.

If

$$F(k^*,\gamma)=2\Phi\left(\sqrt{\gamma}(k^*-\theta_0)\right)-ck^*-1-A(k^*)[c-2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k^*-\theta_0)\right)]>0$$

for some γ, then the proof is complete.

$$\lim_{\gamma\to\infty}F(k^*,\gamma)=-ck^*-1-A(k^*)c>0$$

There exists such a k^* level since, $k^*+A(k^*)<-\frac{1}{c}$ holds for many k^*.
$F(k^*,\gamma)>0$ and $F(\theta_0,\gamma)<0$ along with $\theta_0>0$ imply that there exists at least one $k>k^*$ such that $F(k,\gamma)>0$. Then advertisement level A generates higher market shares.

B. 2 full game equilibria:

Lemma 4. *There exists 2 equilibrium $k's$, for the parameter value γ_0, if γ_0 satisfies the following:*

$$F_k(k_0,\gamma_0)=0 \qquad F(k_0,\gamma_0)=0$$

Proof. The first condition ensures that $k_0=\arg\min_k F(k,\gamma_0)$ or $k_0=\arg\max_k F(k,\gamma_0)$
The second condition chooses the level of γ where $F(k_0,\gamma)$ is tangent to $x-axis$. The following figure illustrates these conditions.

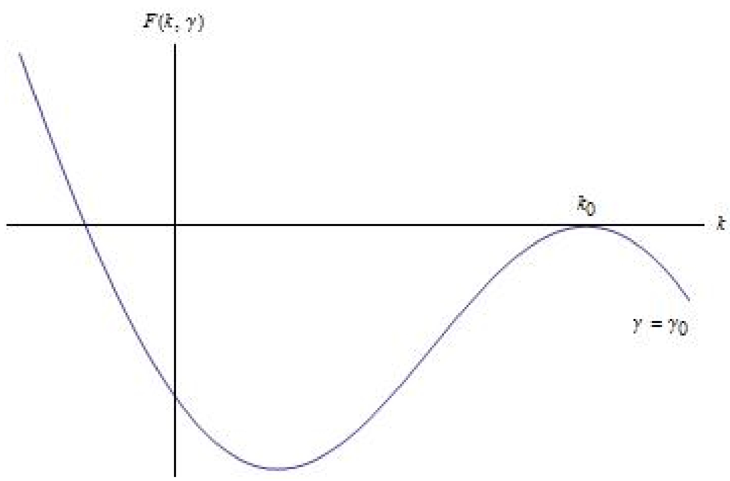

Figure: 2 full game equilibria

Existence of such a γ_0 is followed by the fact that the value function $F(k_0,\gamma)=0$ is continuous in γ and there exist $\gamma's$ where $F(k_0,\gamma)>0$ and $F(k_0,\gamma)<0$. Therefore by intermediate value theorem, there exists a γ_0
Even though there exist multiple equilibria in many of the cases, we can still deduce some properties about the equilibria and parameter values.

C. Properties of the Equilibria:

Lemma 5. *If* $F_\gamma(k_0,\gamma) > 0$ *then,* $F_\gamma(k,\gamma) > 0$ *for all* $\theta_0 < k < k_0$ *if* $k_0 > \theta_0$ *or for all* $k < k_0$ *if* $k_0 < \theta_0$

Proof. Let $F_\gamma(k_0,\gamma) > 0$. Then,

$$F_\gamma(k_0,\gamma) = \gamma^{-1/2}(k_0-\theta_0)\phi\left(\sqrt{\gamma}(k_0-\theta_0)\right) + A(k_0)[\gamma^{-1/2}\phi\left(\sqrt{\gamma}(k_0-\theta_0)\right) + (k_0-\theta_0)\phi'\left(\sqrt{\gamma}(k_0-\theta_0)\right)] > 0$$

Since $A(k_0)(k_0-\theta_0) > 0$. Divide both sides by $A(k_0)\gamma^{-1/2}(k_0-\theta_0)\phi\left(\sqrt{\gamma}(k_0-\theta_0)\right)$ to get,
$\frac{1}{A(k_0)} + \frac{1}{(k_0-\theta_0)} - \gamma(k_0-\theta_0) > 0$
$\frac{1}{A(k_0)} + \frac{1}{(k_0-\theta_0)} > \gamma(k_0-\theta_0)$
For $(k_0-\theta_0) > 0$,

$$\gamma < \frac{1}{A(k_0)(k_0-\theta_0)} + \frac{1}{(k_0-\theta_0)^2} = g(k_0) \text{ with } g'(k) < 0.$$

Let $k_1 < k_0$.
Then, $\gamma < g(k_0) < g(k_1)$.
$F_\gamma(k_1,\gamma) > 0$
For $(k_0-\theta_0) < 0$,
$\gamma > \frac{1}{A(k_0)(k_0-\theta_0)} + \frac{1}{(k_0-\theta_0)^2} = g(k_0)$ with $g'(k) > 0$.
Then, $\gamma > g(k_0) > g(k_1)$
$F_\gamma(k_1,\gamma) > 0$

Lemma 6. *If* $F_\gamma(k_0,\gamma) < 0$ *then,* $F_\gamma(k,\gamma) < 0$ *for all* $\theta_0 > k > k_0$ *if* $k_0 < \theta_0$ *or for all* $k > k_0$ *if* $k_0 > \theta_0$

Proof. Let $F_\gamma(k_0,\gamma) < 0$. Then,
For $(k_0-\theta_0) > 0$,

$$\gamma > \frac{1}{A(k_0)(k_0-\theta_0)} + \frac{1}{(k_0-\theta_0)^2} = g(k_0) \text{ with } g'(k) < 0.$$

Let $k_1 > k_0$.
Then, $\gamma > g(k_0) > g(k_1)$.
$F_\gamma(k_1,\gamma) < 0$
For $(k_0-\theta_0) < 0$,

$$\gamma < \frac{1}{A(k_0)(k_0-\theta_0)} + \frac{1}{(k_0-\theta_0)^2} = g(k_0) \text{ with } g'(k) > 0.$$

Then, $\gamma < g(k_0) < g(k_1)$.
$F_\gamma(k_1,\gamma) < 0$

Lemma 7. *There exists a unique* $k^* > \theta_0$ *where* $F_\gamma(k^*,\gamma) = 0$

Proof. Using the previous two lemmas;
For $k_0 > \theta_0$,
If $F_\gamma(k_0,\gamma) < 0$ then, $F_\gamma(k,\gamma) < 0$ for all $k > k_0$
If $F_\gamma(k_0,\gamma) > 0$ then, $F_\gamma(k,\gamma) > 0$ for all $k < k_0$
Let $k'\&k''$ be such that $F_\gamma(k',\gamma) < 0$ and $F_\gamma(k'',\gamma) > 0$ Then $F_\gamma(k,\gamma) < 0$ for all $k > k'$ and $F_\gamma(k,\gamma) > 0$ for all $k > k''$.
Then, there exists a unique $k^* > \theta_0$ where $F_\gamma(k^*,\gamma) = 0$
At this unique intersection point, the following holds:

$$\gamma = \frac{1}{A(k^*)(k^* - \theta_0)} + \frac{1}{(k^* - \theta_0)^2}$$

or, equivalently, $\sqrt{\gamma}(k^* - \theta_0) = \sqrt{\frac{(k^*-\theta_0)}{A(k^*)} + 1} > 1$
Denote that $\phi\left(\sqrt{\gamma}(k^* - \theta_0)\right)$ is convex if $\sqrt{\gamma}(k^* - \theta_0) > 1$.

D. Optimum Level of Advertisement

The entrant maximizes

$$\max_{A,p_E} \Pi_E = \max_{A,p_E} \lambda p_E - mA^2$$

where $\lambda = 1 - \Phi\left(\frac{k-\theta_0}{\tau}\right)$
Similar to the network externality model, the induced game equilibrium condition is :

$$p_I - p_E = 2\Phi\left(\sqrt{\frac{\gamma}{A}}(k - \theta_0)\right) - 1 - ck$$

Using IFT on equation 3

$$\begin{aligned}
\frac{\partial k}{\partial p_E} &= \frac{1}{c - 2\sqrt{\frac{\gamma}{A}}\phi\left(\sqrt{\frac{\gamma}{A}}(k - \theta_0)\right)} \\
\frac{\partial k}{\partial A} &= \frac{A^{-1}\sqrt{\frac{\gamma}{A}}(k - \theta_0)\phi\left(\sqrt{\frac{\gamma}{A}}(k - \theta_0)\right)}{c - 2\sqrt{\frac{\gamma}{A}}\phi\left(\sqrt{\frac{\gamma}{A}}(k - \theta_0)\right)}
\end{aligned}$$

The First order conditions are

$$\begin{aligned}
\frac{\partial \Pi_E}{\partial A} &= -\frac{p_E}{\tau}\phi\left(\frac{k - \theta_0}{\tau}\right)\frac{\partial k}{\partial A} - 2mA = 0 \\
\frac{\partial \Pi_E}{\partial p_E} &= 1 - \Phi\left(\frac{k - \theta_0}{\tau}\right) - \frac{1}{\tau}p_E\phi\left(\frac{k - \theta_0}{\tau}\right)\frac{\partial k}{\partial p_E} = 0
\end{aligned}$$

The incumbent's problem is the same as in benchmark model, his FOC is the following:

$$\Phi\left(\frac{k - \theta_0}{\tau}\right) = p_I \frac{\phi\left(\frac{k-\theta_0}{\tau}\right)}{\tau[c - 2\sqrt{\gamma}\phi\left(\sqrt{\gamma}(k - \theta_0)\right)]}$$

Therefore, equilibrium condition is:

$$F(k^*, \gamma) = 2\Phi\left(\sqrt{\frac{\gamma}{A}}(k^* - \theta_0)\right) - ck^* - 1 - A(k^*)[c - 2\sqrt{\frac{\gamma}{A}}\phi\left(\sqrt{\frac{\gamma}{A}}(k^* - \theta_0)\right)] = 0$$

$$[1 - \Phi\left(\frac{k - \theta_0}{\tau}\right)]A^{-1}\sqrt{\frac{\gamma}{A}}(k - \theta_0)\phi\left(\sqrt{\frac{\gamma}{A}}(k - \theta_0)\right) + 2mA = 0$$

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